INVESTIGATING p-GROUPS BY COCLASS WITH GAP

Heiko Dietrich* Bettina Eick
Dörte Feichtenschlager*

Technische Universität Braunschweig, Institut Computational Mathematics

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Coclass Graph

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2^2

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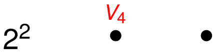

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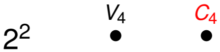

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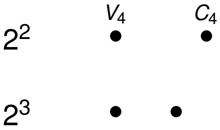

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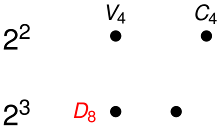

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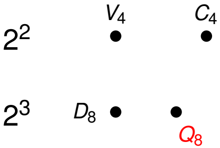

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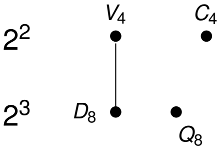

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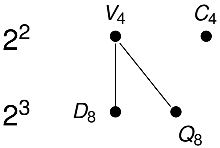

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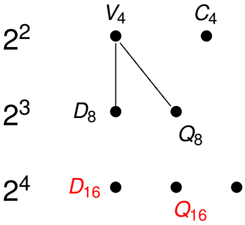

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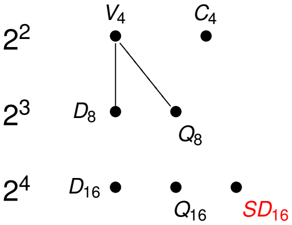

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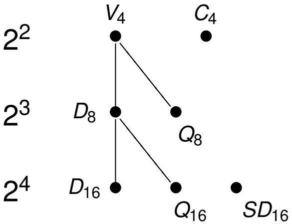

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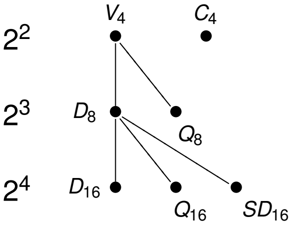

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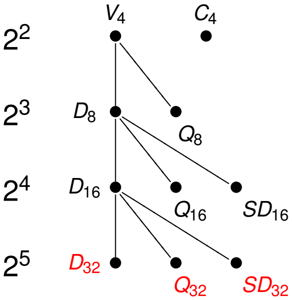

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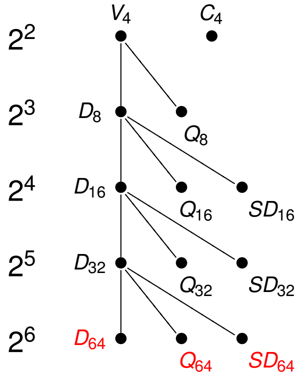

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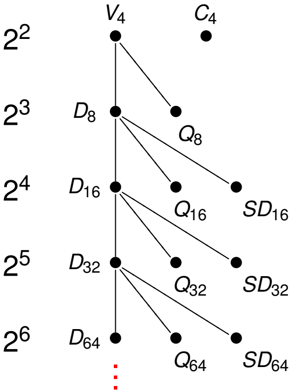

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  - *cohomology rings $H^*(-, R)$, for R ring,*

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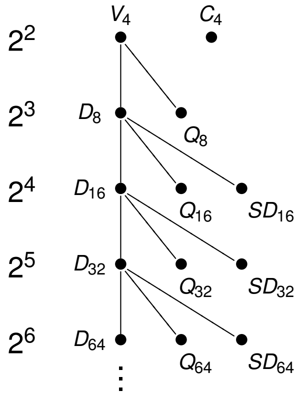

Construction rules and coclass families

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$\mathcal{T}$ coclass tree $\mathcal{G}(p, r)$. $\Rightarrow$ Exist $d, f \in \mathbb{N}$ such that

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THEOREM (EICK, LEEDHAM-GREEN)

Let $\mathcal{T}$ be a bounded coclass tree. Then there exist $d, f \in \mathbb{N}$ and isomorphisms $\mathcal{B}_{i+d} \to \mathcal{B}_i$, $i \geq f$.

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	$\mathcal{T}_1(2,2)$	$\mathcal{T}_2(2,2)$	$\mathcal{T}_3(2,2)$	$\mathcal{T}_4(2,2)$	$\mathcal{T}_5(2,2)$
dim	2	2	1	1	1
G_0	(64,34)	(64,32)	(16,4)	(32,9)	(8,5)
# fam.	19	16	4	6	6
G_f	(64,34)	(64,32)	(16,4)	(32,9)	(16,11)

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- a point group P (finite p-group) acting uniserially on T; i.e. $T_0 = T$ and $T_{i+1} = [T_i, P]$ satisfies $[T_i : T_{i+1}] = p$ for all i.

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Theorem

There is a 1–1 corresp. between the $Aut(G)$-orbits of $\overline{U}$ and the isomorphism types of immediate descendants of G.

Conjecture for cohomology

Theorem (Carslon)

Let k be a field with char$(k) = 2$ and $r \in \mathbb{N}$. Then there exist only finitely many isomorphism types of cohomology rings $H^*(G,k)$ where G is a 2-group of coclass r.

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$\mathcal{G}(2,1)$ and cohomology

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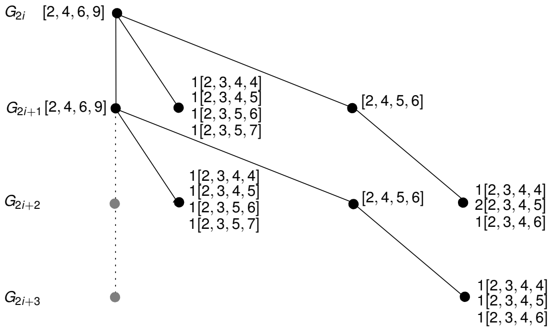
$\mathcal{G}(2,2)$ AND COHOMOLOGY: $\mathcal{T}_1(2,2)$



Conjectured mod2-cohomology for $\mathcal{B}_{2i},\ \mathcal{B}_{2i+1} \subseteq \mathcal{T}_1(2,2)$ $(i \in \mathbb{N}_0)$.

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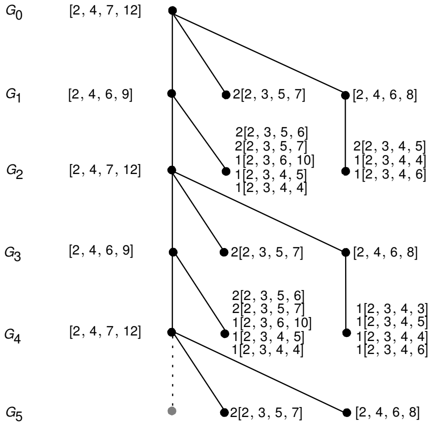

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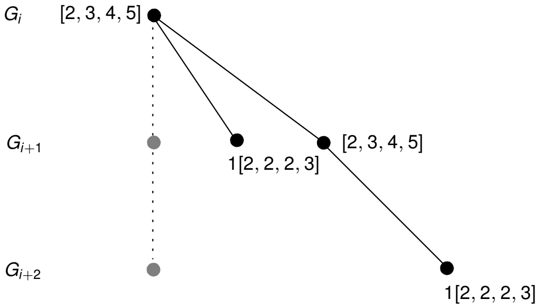
$\mathcal{G}(2,2)$ AND COHOMOLOGY: $\mathcal{T}_3(2,2)$



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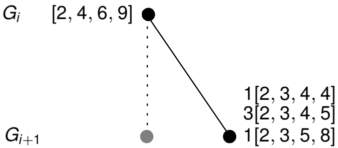
$\mathcal{G}(2,2)$ and cohomology: $\mathcal{T}_4(2,2)$



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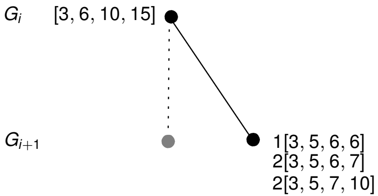
$\mathcal{G}(2,2)$ and cohomology: $\mathcal{T}_5(2,2)$



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$\rightsquigarrow$ Consider $\mathcal{G}(5, 1)$ in more detail.

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It is known:

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THE BRANCH $\mathcal{B}_i$ OF $\mathcal{G}(5, 1)$

CONJECTURE

Let $i \geq 8$ and write $i = 8 + 4x + y$ with $0 \leq y \leq 3$ and $x \geq 0$.

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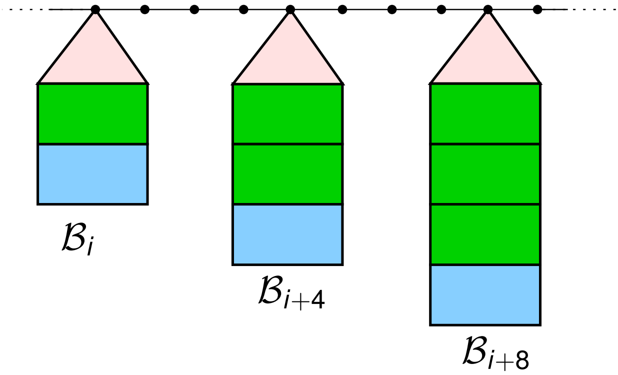
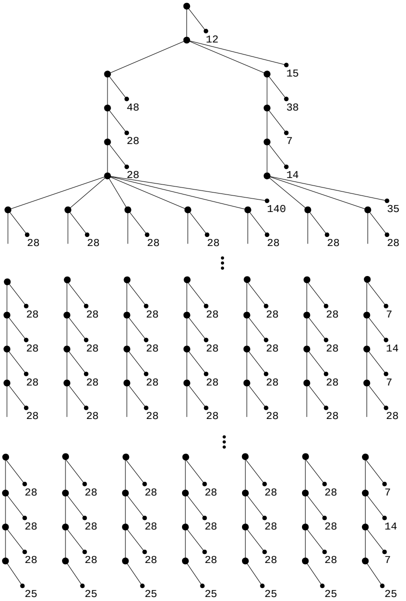
The branches of $\mathcal{G}(5, 1)$

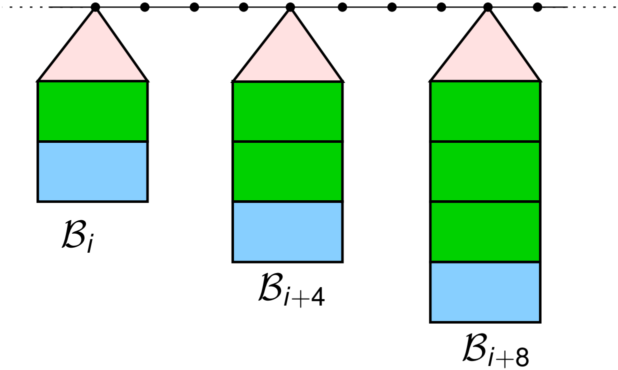


Structures of $\mathcal{B}_i, \mathcal{B}_{i+4}, \ldots$ with $12 \leq i \leq 15$.



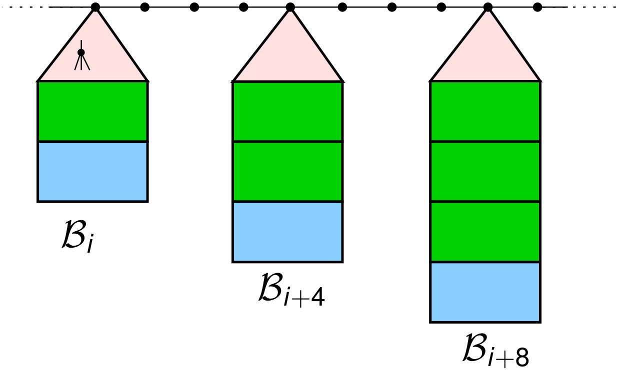
The conjectured branches $\mathcal{B}_i$ with $i = 8 + 4x + 1$ and $x \geq 0$.

$\mathcal{G}(5,1)$ and coclass families



The origins of infinite coclass families in $\mathcal{B}_i$, $12 \leq i \leq 15$.

$\mathcal{G}(5,1)$ AND COCLASS FAMILIES



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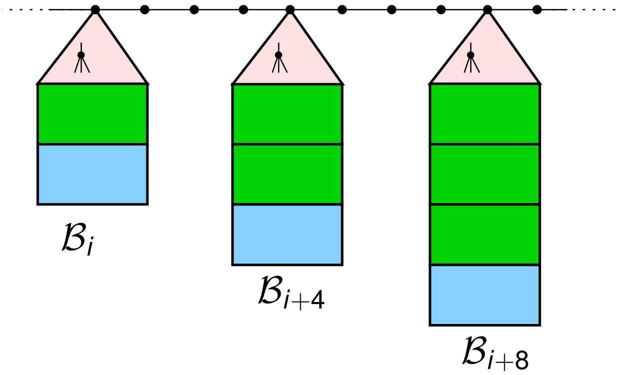

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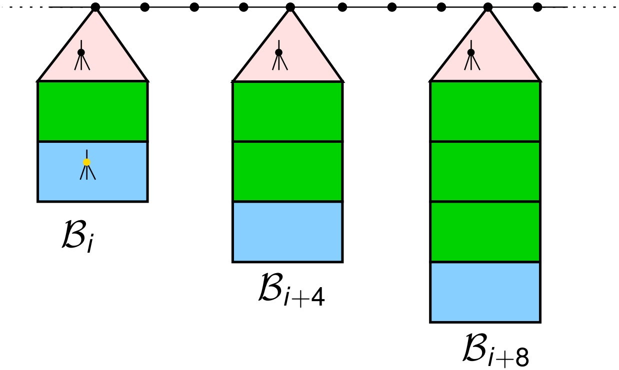

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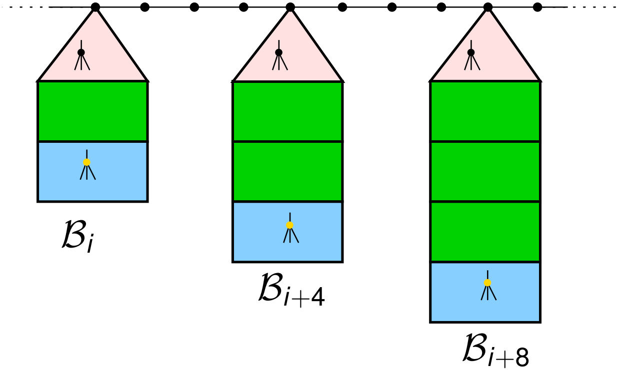

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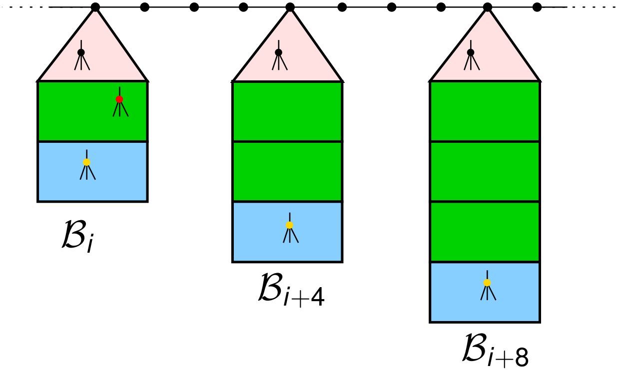

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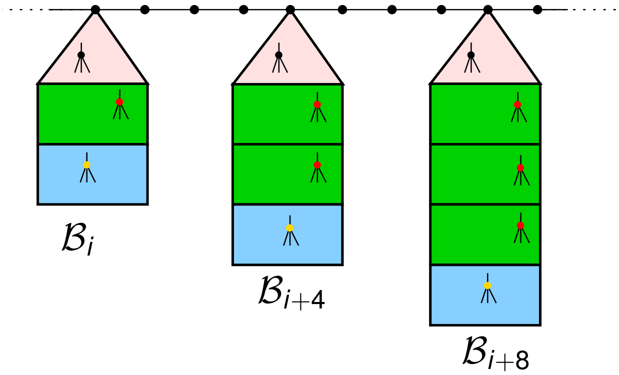

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Infinite coclass families in $\mathcal{G}(5,1)$:

- The groups in $H(i)$, $T(i)$, and $C(i,0)$ with $12 \leq i \leq 15$ would define disjoint infinite coclass families.

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Conjectured number of infinite coclass families:

# families	$y=0$	$y=1$	$y=2$	$y=3$	Σ
in the heads	366	578	741	953	2638
in the collars	748	756	748	756	3008
in the tails	730	735	730	737	2932
Σ	1844	2069	2219	2446	8578

$\mathcal{G}(5,1)$ and Schur multiplicators

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For $n \in \mathbb{N}$ write $n = 4s_n + r_n$ with $1 \leq r_n \leq 4$.

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For $n \in \mathbb{N}$ write $n = 4s_n + r_n$ with $1 \leq r_n \leq 4$.

- *If $G \in \mathcal{B}_i$ with $i \geq 8$ is capable and $|G| = p^n$, then*

$$\mathrm{I(M(G))} = \begin{cases} (5, \quad 5^{s_n}, \quad 5^{s_n}) & \text{if } r_n = 1, 2, \\ (5, \quad 5^{s_n}, \quad 5^{s_n+1}) & \text{if } r_n = 3, 4. \end{cases}$$

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- *If H is a terminal immediate descendant of G, then*

$$\mathrm{I(M(H))} = \begin{cases} (5^{s_n}, \quad 5^{s_n}) & \textit{if } r_n = 1, 2, \\ (5^{s_n}, \quad 5^{s_n+1}) & \textit{if } r_n = 3, 4. \end{cases}$$

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- Let $G \in \mathcal{B}_i$ define a coclass family $\mathcal{F}$.
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Then:

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$$\mathrm{I(M(K))} = \begin{cases} (5^a, 5^{b+l}, 5^{c+l}) & \text{if } G \in H(i), \\ (5^a, 5^{b+2l}, 5^{c+2l}) & \text{if } G \in T(i), \\ (5^a, 5^{b+l+k}, 5^{c+l+k}) & \text{if } G \in C(i,0),\ K \in C(j,k). \end{cases}$$

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Moreover,

- *if $\mathcal{F}$ arises from a head, then $u \in \{1, 2, 4, 16\}$ and $v \in \{-1, 0, 1, 2, 3\}$.*
- *if $\mathcal{F}$ arises from a tail or collar, then $u \in \{1, 2, 4\}$ and $v \in \{2, 3\}$.*

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- Let $p > 3$ be a prime.

  Recall: $S = C_p \ltimes T$ and $G_i = S/\gamma_{i+2}(S)$.

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$\rightsquigarrow$ **Compute Σ-orbits.**

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$\rightsquigarrow$ First step to prove the main conjecture for groups of coclass 1.

Thank you for your kind attention.